A note on capacitors with wide electrode separation

1992 J. Phys. A: Math. Gen. 25 L639

(http://iopscience.iop.org/0305-4470/25/11/003)

J. Phys. A: Math. Gen. **25** (1992) L639–L642. Printed in the UK

LETTER TO THE EDITOR

A note on capacitors with wide electrode separation

H J Wintle

Department of Physics, Queen's University, Kingston, Canada K7L 3N6

Received 19 December 1991

Abstract. We show that for plane parallel capacitors with a large ratio of (area/perimeter^2) and large plate separation, the excess capacitance due to fringing fields decreases as the electrode spacing decreases. This is contrary to intuition, but is supported by numerical work on disc-shaped capacitors.

There has been a revival of interest in the calculation of the capacitance of parallel plate capacitors. This is in part due to the exigencies of accurate measurement [1], but mostly arises because of the use of microstrip circuitry [2]. A pair of capacitor plates mounted opposite one another on the upper and lower faces of an infinite sheet of dielectric of thickness t and dielectric constant ε_2 have a geometric capacitance given by

$$C_{\text{geom}} = \varepsilon_0\varepsilon_2 S/t = \varepsilon_0\varepsilon_1\varepsilon_r S/T \tag{1}$$

where S is the surface area of one plate and ε_0 the permittivity of free space. The surrounding medium has a dielectric constant ε_1 (normally this will be air or vacuum with $\varepsilon_1 = 1$), and $\varepsilon_r = \varepsilon_2/\varepsilon_1$ is the dielectric constant relative to this medium. The actual capacitance C is larger than C_{geom} because of the fringing field, and the excess capacitance (poorly named the edge capacitance in many publications) is defined as

$$C_{\text{ex}} = C - C_{\text{geom}}. \tag{2}$$

It is normally assumed that this excess capacitance increases as the spacing t decreases. One might be tempted to say that this behaviour is intuitively obvious, because as the outer surfaces of the electrodes become closer one might expect more stray flux to pass from one edge region to the other, as shown schematically in figure 1. The purpose of the present letter is to point out that this qualitative assessment is frequently not true. As the separation decreases from infinity, the excess capacitance may in fact fall.

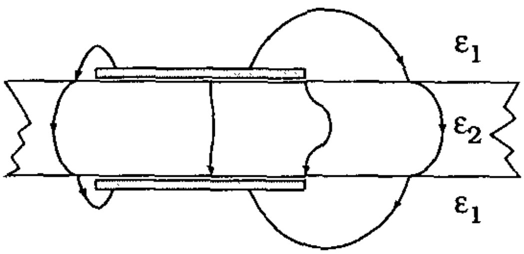

Figure 1. Schematic diagram of a parallel plate capacitor ($\varepsilon_2 > \varepsilon_1$) showing the fringing fields from both the inside faces and the outside faces of the electrodes.

The effect depends in a subtle way on the shape of the electrodes, and on the dielectric constant.

The brute force method of establishing the dependence on plate separation is by direct calculation, as has been done for disc electrodes [1]. Here, the circular symmetry allows one to use a specialized result (a generalization of the Love integral equations) to obtain accurate computations. Square or rectangular geometries can, in principle, be handled by a surface charge simulation method [3], though in a practice it would be difficult to obtain adequate accuracy [4]. The stripline geometry can be evaluated from an analytic expression [5], but other geometries would appear to be quite unamenable to numerical methods.

Fortunately, there is a subtle but general analytic result which relates the capacitance at large separation to the capacitance C_∞ of a single isolated electrode in a vacuum [2]. Strictly, we should specify that the second electrode is a conducting sphere of infinite radius. This relation is

$$C = \frac{(\varepsilon_r + 1) C_\infty \varepsilon_1 / 4}{1 - C_\infty / 2\pi\varepsilon_0 t f(\varepsilon_r)} \qquad t \to \infty \tag{3}$$

with

$$f(\varepsilon_r) = \frac{\varepsilon_r - 1}{\varepsilon_r \ln(2\varepsilon_r / (\varepsilon_r + 1))}. \tag{4}$$

We note that equation (3) differs from equation (2.12) of [2] in several details. The present version refers to a pair of identical electrodes, rather than to one finite electrode and an infinite ground plane. Consequently our thickness t is double the quantity in the original equation while the capacitance is halved, so the numerical factors in both numerator and denominator have been adjusted accordingly. In addition, the original equation was based on the assumption $\varepsilon_1 = 1$, while here we allow for the general case. Since the term $(C_\infty / 2\pi\varepsilon_0 t f)$ in the denominator is a purely geometric factor, it does not need an ε_1 multiplier.

We now look at the gradient G of the excess capacitance with respect to the reciprocal of the separation. It is convenient to introduce a scaling length l and to deal with dimensionless quantities. This leads to

$$\begin{aligned} G &= \frac{\partial}{\partial(l/t)} \left[\frac{C_{\mathrm{ex}}}{\varepsilon_0 \varepsilon_1 \varepsilon_r l} \right] \\ &= \left(\frac{\varepsilon_r + 1}{4} \right) \left(\frac{C_\infty}{\varepsilon_0 \varepsilon_r l} \right) \left(\frac{C_\infty}{2\pi\varepsilon_0 f l} \right) - \frac{S}{l^2} \qquad t \to \infty. \end{aligned} \tag{5}$$

To begin with, we concentrate on finding the sign of the gradient. We need values for C_∞. These are not easily found for arbitrary electrode shapes, but we can make some progress by putting in known lower and upper bounds. For the lower bound we have [6]

$$C_\infty \geqslant 8\varepsilon_0 \sqrt{S/\pi} \tag{6}$$

and by choosing $l = \sqrt{S}$, we obtain the following lower bound G_l for the gradient

$$G_l = \frac{8}{\pi^2} \frac{1}{f} \frac{\varepsilon_r + 1}{\varepsilon_r} - 1 \qquad l = \sqrt{S}. \tag{7}$$

This quantity is negative for all $\varepsilon_r > 1$.

An upper bound G_u for the gradient can be found using the result [7]

$$C_\infty \leqslant 4\pi\varepsilon_0(L/\pi^2) \tag{8}$$

where L is the perimeter of the electrode. If we set $l = L$, we now obtain

$$G_u = \frac{2}{\pi^3}\left(\frac{\varepsilon_r+1}{\varepsilon_r}\right)\frac{1}{f} - \frac{S}{L^2} \qquad l = L. \tag{9}$$

When $\varepsilon_r \geqslant 1$, this upper bound is negative for all regular polygons with five or more sides. For square electrodes it switches sign from slightly positive to negative, while for strip electrodes and other long, thin figures for which $S/L^2 \to 0$ it is clearly positive. Thus we conclude that for all relatively compact electrode shapes, the excess capacitance falls as the separation is reduced from infinity.

The disc capacitor is especially easy to deal with because the capacitance of an isolated disc is

$$C_\infty = 8\varepsilon_0 a \tag{10}$$

where a is the radius of the disc. The equality signs are satisfied in equations (6) and (8), and from (5) we have the exact result

$$G = \frac{8}{\pi}\left(\frac{\varepsilon_r+1}{\varepsilon_r}\right)\frac{1}{f} - \pi \qquad (\text{disc}, l = a). \tag{11}$$

This result can also be established by executing the first Picard iteration of the relevant integral equation ([1] equation (18)), and picking out the coefficient of the term in (a/t) in the limit $(a/t) \to 0$ to give the first member on the right-hand side.

Accurate numerical values are available for the disc case [1]. In figure 2 we have plotted the dimensionless capacitance $(C_{ex}/\varepsilon_0\varepsilon_1\varepsilon_r a)$ against the reciprocal of the dimensionless separation (a/t) for wide separations, with ε_r as the parameter. The

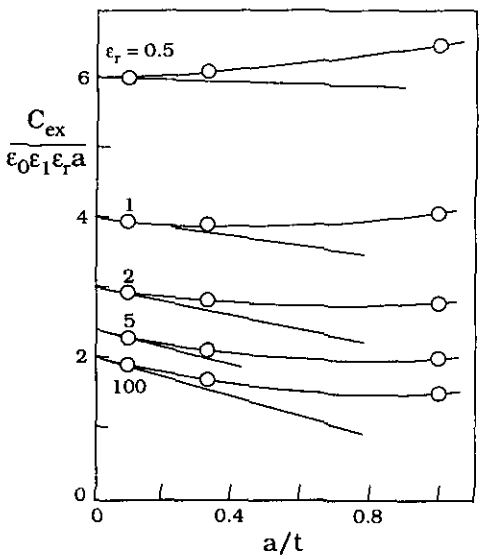

Figure 2. Plot of the excess capacitance (in dimensionless form) against the inverse separation for a disc capacitor. Points are calculated [1], the solid lines show the initial slope from equation (11) and the parameter is the relative dielectric constant ε_r.

initial slopes given by (11) are also plotted, and are seen to be in excellent agreement with the computed data. While it would appear that the second derivative of any one of the curves is zero as $(a/t)\to 0$, the numerical data is too sparse to confirm this conjecture, while the analytical development [2] is only first order in (a/t), and so sheds no light on this possibility.

At the other extreme, we can deal with the stripline geometry. In the limit $\varepsilon_r\to\infty$, the capacitance of a pair of strip electrodes of breadth a, length l, can be expressed in closed form [8],

$$C/\varepsilon_0\varepsilon_2 l = K'(k)/2K(k) \tag{12}$$

$$k = \exp(-\pi a/t) \tag{13}$$

where $K(k)$ and $K'(k)$ are the complete elliptic integrals of the first kind, and end effects are ignored. With a wide separation we have $k\to 1$, $K'\to\pi/2$ and $K\to \frac{1}{2}\ln(8t/\pi a)$. It is then easy to show that the gradient G becomes $+\infty$ as the width shrinks to zero. The situation for $\varepsilon_r = 1$ can be written in a somewhat similar way [8], but in this case, the argument k is no longer a simple function of the geometric ratio, and we must resort to somewhat tedious expansions of the analytic expressions [5, 9, 10]. These lead to the asymptotic behaviour

$$C/\pi\varepsilon_0 l \simeq -1/\ln(C_g/\varepsilon_0 l) \tag{14}$$

a result that is confirmed by numerical methods. Straightforward calculations then indicate that in this case as well, the gradient becomes infinite. We infer that the same will also be true for intermediate values of ε_r.

In summary, we have established that the excess (or edge) capacitance of a parallel plate system does indeed fall as the spacing is reduced from infinity, so long as the plates have a large enough ratio of (area/perimeter^2). This is unexpected, but had been implicitly predicted analytically [2]. We have shown that existing numerical results [1] fully support the theoretical forecast. Other recent work on disc capacitors [11–13] deals with the more conventional case of closely spaced electrodes, so unfortunately it does not provide further confirmation.

I wish to thank Dr E F Kuester (University of Colorado) for some helpful discussion, and NSERC (Canada) for continued financial support.

References

[1] Wintle H J and Goad D G W 1989 *J. Phys. D: Appl. Phys.* **22** 1620–6
[2] Kuester E F 1987 *J. Electromag. Waves Appl.* **2** 103–35
[3] Fuller J A and Chang D C 1970 *Proc. IEEE* **58** 490–1
[4] Wintle H J 1990 *Proc. 23rd Symp. Electrical Insulating Materials* IEEJ pp 23–35 and *J. Electrostat.* in press
[5] Wintle H J and Kurylowicz S 1985 *IEEE Trans. Instrum. Meas.* **34** 41–7
[6] Polya G and Szego G 1951 *Isoperimetric Inequalities in Mathematical Physics* (Princeton, NJ: Princeton University Press)
[7] Kazantsev V P 1983 *Zh. Tekh. Fiz.* **53** 458–65 (Eng. transl. *Sov. Phys.-Tech. Phys.* **8** 287–90 1983)
[8] Smythe W R 1967 *Static and Dynamic Electricity* 3rd edn (New York: McGraw-Hill) p 114, prob 61, p 275, prob 14 and p 115, prob 66
[9] Love A E H 1924 *Proc. London Math. Soc.* **22** 337–69
[10] Kober H 1957 *Dictionary of Conformal Representations* 2nd edn (New York: Dover) pp 194–5
[11] Bartlett D F and Corle T R 1985 *J. Phys. A: Math. Gen.* **18** 1337–42
[12] Sloggett G J, Barton N G and Spencer S J 1986 *J. Phys. A: Math. Gen.* **19** 2725–36
[13] Sloggett G J, Barton N G and Spencer S J 1987 *J. Phys. A: Math. Gen.* **20** 4061–2